

ELECTRIC-MAGNETIC DUALITY ROTATIONS IN NON-LINEAR ELECTRODYNAMICS

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Abstract

We show that there is a function of one variable's worth of Lagrangians for a single Maxwell field coupled to gravity whose equations of motion admit electric-magnetic duality. Such Lagrangians are given by solutions of the Hamilton-Jacobi equation for timelike geodesics in Wittten's two-dimensional black hole. Among them are the Born-Infeld Lagrangian which arises in open string theory. We investigate the effect of the axion and the dilaton in the open superstring case and we show that this theory loses its electric-magnetic duality invariance when one considers the higher order electromagnetic field terms. We discuss some implications for black holes in string theory and an extension to $2k$ -forms in $4k$ spacetime dimensions.

1 Introduction

In ordinary Maxwell theory in Minkowski spacetime a Hodge duality rotation is an action of $SO(2)$

$$\begin{cases} \mathbf{E} \rightarrow \cos \alpha \mathbf{E} - \sin \alpha \mathbf{B} \\ \mathbf{B} \rightarrow \cos \alpha \mathbf{B} + \sin \alpha \mathbf{E} \end{cases} \quad (1.1)$$

which takes solutions of the sourceless Maxwell's equations into solutions and which moreover commutes with Lorentz transformations. If one writes the duality transformation in the form

$$F_{\mu\nu} \rightarrow \cos \alpha F_{\mu\nu} + \sin \alpha \star F_{\mu\nu} \quad (1.2)$$

where \star denotes the Hodge star operation one sees that invariance under duality rotations continues to be a symmetry of Maxwell's equations in a curved spacetime. One can ask whether a generalization of duality invariance continues to hold if one modifies the Maxwell action. In particular one may consider Lagrangian densities $\mathcal{L} = \sqrt{-g} L$ depending only on a single Maxwell field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the spacetime metric $g_{\alpha\beta}$ but which are not quadratic in the two-form $F_{\mu\nu}$ and whose equations are therefore non-linear. The purpose of this paper is to answer this question. We shall show that there are as many such Lagrangians as there are solutions of the Hamilton-Jacobi equation for timelike geodesics in a two dimensional Witten black hole spacetime [1]. Thus, roughly speaking, such Lagrangians depend upon an arbitrary function of a single real variable. A particular example is the Born-Infeld Lagrangian [2]

$$\mathcal{L} = \frac{1}{b^2} \left\{ \sqrt{-g} - \sqrt{-\det(g_{\mu\nu} + bF_{\mu\nu})} \right\}, \quad (1.3)$$

where the constant b has the dimensions of length squared.¹ Of course if one keeps the Lagrangian quadratic in $F_{\mu\nu}$ but couples extra fields, such as an axion and dilaton, it is known [3] that one may extend the $SO(2)$ invariance to an $SL(2, \mathbb{R})$ invariance but as far as we are aware all existing discussions

¹In this paper we use units in which $c = \hbar = \varepsilon_0 = \mu_0 = 1$, so that the parameter b , Newton's constant G and the inverse string tension α' all have the dimensions of length squared. In these units, electric and magnetic charges are dimensionless.

of duality do not consider equations of motion which are non-linear in the Maxwell field. Because this case arises in open string theory [4], with

$$b = 2\pi\alpha', \quad (1.4)$$

we consider it worth a special investigation. In particular the Born-Infeld Lagrangian arises in open superstring theory, together with an axion and a dilaton, and we shall show later that, in this case, all electric-magnetic duality is lost.

In what follows we shall adopt the following conventions, $\eta_{\mu\nu\lambda\rho}$ will be the covariantly constant volume form and indices will be freely raised and lowered using the metric $g_{\mu\nu}$ whose signature is $-+++$. Thus the Hodge star operation is given by $F_{\mu\nu} \rightarrow \star F_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu}^{\alpha\beta}F_{\alpha\beta}$ and we have $\star\star = -1$. The electric intensity \mathbf{E} and magnetic induction \mathbf{B} are defined in a local orthonormal frame by $E_i = F_{i0}$ and $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$. One has $\star\mathbf{E} = -\mathbf{B}$ and $\star\mathbf{B} = \mathbf{E}$. Therefore $F_{\alpha\beta}F^{\alpha\beta} = 2(\mathbf{B}^2 - \mathbf{E}^2)$ and $F_{\alpha\beta}\star F^{\alpha\beta} = 4\mathbf{E} \cdot \mathbf{B}$. The Bianchi identities are

$$\partial_{[\alpha}F_{\beta\gamma]} = 0, \quad (1.5)$$

which are equivalent to

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}. \end{aligned} \quad (1.6)$$

Given a Lagrangian L one may define $G^{\mu\nu}$ by ²

$$G^{\mu\nu} = -2\frac{\partial L}{\partial F_{\mu\nu}}. \quad (1.7)$$

The field equations are

$$\partial_{[\alpha}\star G_{\beta\gamma]} = 0. \quad (1.8)$$

If one defines the electric induction \mathbf{D} and magnetic intensity \mathbf{H} by $D_i = G_{i0}$ and $H_i = \frac{1}{2}\epsilon_{ijk}G_{jk}$, then the field equations are equivalent in Minkowski

²There is some ambiguity in the definition of this partial derivative depending on whether or not one takes into account the antisymmetry of $F_{\mu\nu}$. Here we treat $F_{\mu\nu}$ and $F_{\nu\mu}$ as independent variables, hence the factor of 2.

spacetime to

$$\begin{aligned}\nabla \cdot \mathbf{D} &= 0 \\ \nabla \times \mathbf{H} &= +\frac{\partial \mathbf{D}}{\partial t}.\end{aligned}\tag{1.9}$$

The field equations and Bianchi identities may be combined in the form

$$\begin{aligned}\nabla \cdot (\mathbf{D} + i\mathbf{B}) &= 0 \\ \nabla \times (\mathbf{E} + i\mathbf{H}) &= i\frac{\partial}{\partial t}(\mathbf{D} + i\mathbf{B}).\end{aligned}\tag{1.10}$$

Thus the generally covariant generalization of Hodge duality rotations to a general electric-magnetic duality rotation through an angle α is

$$\begin{cases} F_{\mu\nu} \rightarrow \cos \alpha F_{\mu\nu} + \sin \alpha \star G_{\mu\nu} \\ G_{\mu\nu} \rightarrow \cos \alpha G_{\mu\nu} + \sin \alpha \star F_{\mu\nu}. \end{cases}\tag{1.11}$$

In Minkowski spacetime this is equivalent to

$$\begin{cases} \mathbf{E} + i\mathbf{H} \rightarrow e^{i\alpha}(\mathbf{E} + i\mathbf{H}) \\ \mathbf{D} + i\mathbf{B} \rightarrow e^{i\alpha}(\mathbf{D} + i\mathbf{B}). \end{cases}\tag{1.12}$$

It is clear that if $F_{\mu\nu}$ and $G_{\mu\nu}$ were independent variables then electric-magnetic duality rotations would take the linear field equations into the linear Bianchi identities and conversely regardless of the specific form of the Lagrangian L . However they are not independent of one another; they are linked by the constitutive relation $G^{\mu\nu} = -2\frac{\partial L}{\partial F_{\mu\nu}}$ which in general is a non-linear relation between $G_{\mu\nu}$ and $F_{\mu\nu}$. Only for a restricted class of Lagrangians will this relation be invariant under electric-magnetic duality rotations. Note that one should distinguish electric-magnetic duality rotations from Hodge rotations. Hodge rotations transform \mathbf{E} into \mathbf{B} and \mathbf{D} into \mathbf{H} . Except for the case of Maxwell theory, for which $F_{\mu\nu} = G_{\mu\nu}$ and hence $\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$, Hodge rotations do not take the Bianchi identities into the equations of motion.

2 Duality Invariance

In order to find which Lagrangians admit electric-magnetic duality rotations it suffices to consider infinitesimal transformations which take the form

$$\begin{cases} \delta F_{\mu\nu} = \star G_{\mu\nu} \\ \delta G_{\mu\nu} = \star F_{\mu\nu}. \end{cases} \quad (2.1)$$

The invariance of the constitutive relation under an infinitesimal duality rotation requires that

$$\frac{1}{2}\eta^{\mu\nu\lambda\rho}F_{\lambda\rho} = \frac{1}{2}\eta_{\sigma\tau\alpha\beta}G^{\alpha\beta}\frac{\partial}{\partial F_{\sigma\tau}}\left(-2\frac{\partial L}{\partial F_{\mu\nu}}\right). \quad (2.2)$$

Substituting the definition of $G_{\alpha\beta}$ and using the commutation of partial derivatives gives

$$\frac{1}{2}\eta^{\mu\nu\lambda\rho}F_{\lambda\rho} = 2\eta_{\sigma\tau\alpha\beta}\frac{\partial L}{\partial F_{\alpha\beta}}\frac{\partial}{\partial F_{\mu\nu}}\left(\frac{\partial L}{\partial F_{\sigma\tau}}\right). \quad (2.3)$$

This second order partial differential equation in the six variables $F_{\mu\nu}$ is the necessary and sufficient condition on the Lagrangian L that its Euler-Lagrange equations admit duality invariance. Because $\eta_{\alpha\beta\sigma\tau} = \eta_{\sigma\tau\alpha\beta}$ we have

$$\frac{1}{2}\eta^{\mu\nu\lambda\rho}F_{\lambda\rho} = \frac{\partial}{\partial F_{\mu\nu}}\left(\eta_{\sigma\tau\alpha\beta}\frac{\partial L}{\partial F_{\sigma\tau}}\frac{\partial L}{\partial F_{\alpha\beta}}\right). \quad (2.4)$$

A first integral of our second order partial differential equation is obtained by integrating with respect to $F_{\mu\nu}$, remembering again that $\eta_{\alpha\beta\sigma\tau} = \eta_{\sigma\tau\alpha\beta}$. This gives the first order partial differential equation :

$$\frac{1}{4}\eta^{\mu\nu\lambda\rho}F_{\mu\nu}F_{\lambda\rho} = \eta_{\sigma\tau\alpha\beta}\frac{\partial L}{\partial F_{\sigma\tau}}\frac{\partial L}{\partial F_{\alpha\beta}} + 2C, \quad (2.5)$$

where C is an arbitrary constant of integration. In fact, if L is to agree with the usual Maxwell Lagrangian at weak fields, the constant must vanish but we shall retain it for the time being. Note that the condition we have obtained is manifestly Lorentz-invariant. By differentiating the first integral one sees that every such first integral satisfies the second order equation provided the constitutive relation holds.

Using the definition of $G_{\mu\nu}$ we may write the first integral as

$$F_{\mu\nu} \star F^{\mu\nu} = G_{\mu\nu} \star G^{\mu\nu} + 4C. \quad (2.6)$$

In terms of the electric and magnetic intensities \mathbf{E} and \mathbf{H} and the electric and magnetic inductions \mathbf{D} and \mathbf{B} our condition reads

$$\mathbf{E} \cdot \mathbf{B} = \mathbf{D} \cdot \mathbf{H} + C. \quad (2.7)$$

Equation (2.6) has an important consequence for the duality transformation properties of the energy-momentum tensor, which is defined by

$$\sqrt{-g} T_{\mu\nu} = -2 \frac{\partial}{\partial g^{\mu\nu}} (\sqrt{-g} L), \quad (2.8)$$

where the partial derivative with respect to $g_{\mu\nu}$ is taken with $F_{\mu\nu}$ held fixed. Under an infinitesimal duality rotation $T_{\mu\nu}$ will therefore transform according to

$$\begin{aligned} \sqrt{-g} \delta T_{\mu\nu} &= -2 \frac{\partial}{\partial g^{\mu\nu}} (\sqrt{-g} \delta L) \\ &= -2 \frac{\partial}{\partial g^{\mu\nu}} \left(\sqrt{-g} \frac{\partial L}{\partial F_{\lambda\rho}} \star G_{\lambda\rho} \right) \\ &= \frac{\partial}{\partial g^{\mu\nu}} \left(\sqrt{-g} G^{\lambda\rho} \star G_{\lambda\rho} \right). \end{aligned} \quad (2.9)$$

If equation (2.6) holds then

$$\begin{aligned} \sqrt{-g} \delta T_{\mu\nu} &= \frac{\partial}{\partial g^{\mu\nu}} \left(\sqrt{-g} F^{\lambda\rho} \star F_{\lambda\rho} - 4C\sqrt{-g} \right) \\ &= \frac{1}{2} F_{\lambda\rho} F_{\sigma\kappa} \frac{\partial}{\partial g^{\mu\nu}} \left(\sqrt{-g} \eta^{\lambda\rho\sigma\kappa} \right) + 2Cg_{\mu\nu}\sqrt{-g} \\ &= 2Cg_{\mu\nu}\sqrt{-g}. \end{aligned} \quad (2.10)$$

Thus the energy-momentum tensor of a duality invariant theory with $C = 0$ is itself invariant under duality rotations.

3 Hamiltonian Viewpoint

The condition that we have obtained has an interesting Hamiltonian geometric interpretation which is useful in discussing its solution and which also applies to more complicated models. The six-dimensional space $V = \Lambda^2(\mathbb{R}^4)$ of two-forms in \mathbb{R}^4 , has coordinates $F_{\mu\nu}$ and carries a Lorentz-invariant metric k with signature $+++--$ defined by

$$k(F, F) = \frac{1}{2}\eta^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} = 4\mathbf{E} \cdot \mathbf{B}. \quad (3.1)$$

The dual space V^* of V consists of skew-symmetric second rank contravariant tensors $G^{\mu\nu}$. The phase space $\mathcal{P} = V \oplus V^*$ carries a natural symplectic structure :

$$\frac{1}{2}dG^{\mu\nu} \wedge dF_{\mu\nu} = d\mathbf{B} \wedge \cdot d\mathbf{H} - d\mathbf{D} \wedge \cdot d\mathbf{E}. \quad (3.2)$$

The constitutive relation connecting $G^{\mu\nu}$ and $F_{\mu\nu}$ is a Legendre transformation with generating function L and defines a Lagrangian submanifold \mathcal{L} of the phase space $\mathcal{P} = V \oplus V^*$.

Electric-magnetic duality rotations act symplectically on \mathcal{P} with generating function or moment map $K : \mathcal{P} \rightarrow \mathbb{R}$ given by

$$K = \frac{1}{2}(G_{\mu\nu} \star G^{\mu\nu} - F_{\mu\nu} \star F^{\mu\nu}). \quad (3.3)$$

The Legendre transformation will therefore commute with electric-magnetic duality rotations if the Lagrangian submanifold \mathcal{L} is invariant under the action of electric-magnetic duality rotations.

Now it is true quite generally that a Lagrangian submanifold $p_i = \frac{\partial S}{\partial q^i}$, with generating function $S(q^i)$, of a symplectic manifold \mathcal{P} with canonical coordinates p_i, q^i , is invariant under the Hamiltonian flow generated by a moment map $K(p_i, q^i)$ if and only if S satisfies the Hamilton-Jacobi equation associated to the Hamiltonian function $K(p_i, q^i)$, that is

$$K\left(\frac{\partial S}{\partial q^i}, q^i\right) = \kappa, \quad (3.4)$$

where κ is a constant labelling the level sets of the Hamiltonian K on which the flow lies. To obtain the present case we set $S(q^i) = L(F_{\mu\nu})$ and $\kappa = -2C$.

One may, if one wishes, regard L as satisfying the Hamilton-Jacobi equation associated to geodesic flow with respect to the Jacobi-metric h on $V = \Lambda^2(\mathbb{R}^4)$ given by

$$h(F, F) = \frac{1}{2} \frac{dF_{\mu\nu} \otimes \star dF^{\mu\nu}}{F_{\alpha\beta} \star F^{\alpha\beta} - 4C}. \quad (3.5)$$

For a Lorentz-invariant theory the Lagrangian function $L = L(F_{\mu\nu})$ can only depend on the two independent Lorentz scalars $F_{\alpha\beta} F^{\alpha\beta} = 2(\mathbf{B}^2 - \mathbf{E}^2)$ and $F_{\alpha\beta} \star F^{\alpha\beta} = 4\mathbf{E} \cdot \mathbf{B}$. Thus our condition effectively reduces to a partial differential equation in two variables. It turns out to be most convenient to make use of the freedom to make Lorentz-transformations to pass to a frame in which \mathbf{E} and \mathbf{B} are parallel. It then follows that both \mathbf{D} and \mathbf{H} will also be parallel to both \mathbf{E} and \mathbf{B} . This is possible everywhere in V except on the “lightcone” of the metric k where the invariant $F_{\alpha\beta} \star F^{\alpha\beta} = 4\mathbf{E} \cdot \mathbf{B}$ vanishes. We therefore introduce the two variables given by

$$\begin{aligned} E^2 &= \frac{1}{2} \left\{ \mathbf{E}^2 - \mathbf{B}^2 + \sqrt{(\mathbf{B}^2 - \mathbf{E}^2)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2} \right\} \\ B^2 &= \frac{1}{2} \left\{ \mathbf{B}^2 - \mathbf{E}^2 + \sqrt{(\mathbf{B}^2 - \mathbf{E}^2)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2} \right\} \end{aligned} \quad (3.6)$$

so that E and B are the magnitudes of \mathbf{E} and \mathbf{B} in this special frame. In terms of E and B the Hamilton-Jacobi equation becomes

$$\frac{\partial L}{\partial E} \frac{\partial L}{\partial B} = C - EB. \quad (3.7)$$

This is the Hamilton-Jacobi equation for geodesics in the two-dimensional metric

$$ds^2 = \frac{dEdB}{C - EB}. \quad (3.8)$$

Interestingly E and B are null coordinates for Witten’s two dimensional black hole spacetime W [1]. The general solution of this equation may be constructed by picking an initial spacelike curve Σ in W on which L takes some constant value. The value of L at a point in $p = (E, B) \in W$ not on Σ is then the action or proper time along a timelike geodesic joining p to Σ . Since there may be more than one such geodesic the Lagrangian L will in

general have branch points and be multi-valued. The initial curve Σ may be specified by giving an arbitrary function of a single variable. Alternatively one may fix a Cauchy curve Σ on which L is allowed to take arbitrary values. L is then obtained at a point $p \in W$ not on Σ by joining p to Σ by a timelike geodesic of proper time $\tau(p)$ meeting Σ in the point $q(p)$. One then has

$$L(p) = L(q(p)) + \tau(p). \quad (3.9)$$

Convenient initial Cauchy surfaces are the null surfaces $B = 0$ or $E = 0$. Thus for example one may specify arbitrarily the constitutive relation $D = \frac{\partial L}{\partial E}$ at zero magnetic field or $H = -\frac{\partial L}{\partial B}$ at zero electric field and extend it to non-vanishing magnetic or electric fields respectively in such a way that it is duality invariant. Thus there are, roughly speaking, as many electric-magnetic duality invariant generalizations of Maxwell's equations as there are functions of a single variable. Physically one presumably wants them to coincide with Maxwell's theory at small values of E and B in which case a necessary condition is that $C = 0$. If $C = 0$ and

$$u = \frac{1}{2}E^2 \quad , \quad v = \frac{1}{2}B^2, \quad (3.10)$$

then the Hamilton-Jacobi equation becomes even simpler; it reduces to that for timelike geodesics in two-dimensional Minkowski spacetime

$$\frac{\partial L}{\partial u} \frac{\partial L}{\partial v} = -1. \quad (3.11)$$

4 Duality Invariant Lagrangians

To obtain explicit solutions of (3.11) one must resort to techniques such as separation of variables in particular coordinate systems. For example if one supposes that the solution separates multiplicatively in (u, v) coordinates one obtains

$$L = \pm \sqrt{\alpha - \beta E^2} \sqrt{\gamma - \delta B^2} \quad , \quad \beta\delta = 1. \quad (4.1)$$

For general $\alpha, \beta, \gamma, \delta$ this does not coincide with Maxwell theory for small values of E and B , the terms quadratic in E and B being different from those in Maxwell theory, so the condition $C = 0$ is not sufficient to ensure that the theory has the correct weak field limit. It can be shown that the

only Lagrangian of the form (4.1) which does have the correct weak field limit is given by $\delta = 1/\beta$, $\gamma = \alpha/\beta^2$, $\beta = \alpha b^2$ which is equivalent to the Born-Infeld Lagrangian

$$\begin{aligned} L &= \frac{1}{b^2} \left\{ 1 - \sqrt{1 + b^2 (\mathbf{B}^2 - \mathbf{E}^2) - b^4 (\mathbf{E} \cdot \mathbf{B})^2} \right\} \\ &= \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) + \frac{1}{8} b^2 (\mathbf{E}^2 - \mathbf{B}^2)^2 + \frac{1}{2} b^2 (\mathbf{E} \cdot \mathbf{B})^2 + \mathcal{O}(6), \end{aligned} \quad (4.2)$$

where $\mathcal{O}(6)$ denotes terms of order 6 in \mathbf{E} and \mathbf{B} . The fact that Born-Infeld theory is electric-magnetic duality invariant seems to have been noticed first by Schrödinger [5]. It is interesting to note that the Euler-Heisenberg Lagrangian [6] for a supersymmetric system of minimally coupled spin- $\frac{1}{2}$ and spin-0 particles agrees with the Born-Infeld Lagrangian, with

$$b^2 = \frac{e^4}{24\pi^2 m^4}, \quad (4.3)$$

up to and including terms $\mathcal{O}(4)$, and hence it is duality invariant to that order. This may indicate that, under some circumstances, classical duality may persist at the quantum level.

Approximate solutions to equation (3.11) can also be obtained by a power series expansion. It is convenient to define new variables x and y by

$$\begin{aligned} x = u + v &= \frac{1}{2} (E^2 + B^2) \\ y = u - v &= \frac{1}{2} (E^2 - B^2). \end{aligned} \quad (4.4)$$

Then (3.11) becomes

$$\left(\frac{\partial L}{\partial x} \right)^2 - \left(\frac{\partial L}{\partial y} \right)^2 = -1. \quad (4.5)$$

Assuming that L coincides with the Maxwell Lagrangian for small values of E and B , i.e. $L = y + \mathcal{O}(x^2, y^2)$, then it is found that the quadratic terms in x and y must be proportional to x^2 , i.e.

$$\begin{aligned} L &= y + ax^2 + \mathcal{O}(x^3, y^3) \\ &= \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) + \frac{1}{4} a (\mathbf{E}^2 - \mathbf{B}^2)^2 + a (\mathbf{E} \cdot \mathbf{B})^2 + \mathcal{O}(6). \end{aligned} \quad (4.6)$$

This coincides with the Born-Infeld Lagrangian up to and including terms quartic in \mathbf{E} and \mathbf{B} , on setting $a = \frac{1}{2}b^2$.

Equation (4.5) is insufficient to determine uniquely the higher order terms in the expansion of L . Writing L in the form

$$L = y + f(x) + \sum_{n=3}^{\infty} L^{(n)}(x, y), \quad (4.7)$$

where f is an arbitrary function of x satisfying $f(0) = f'(0) = 0$ and each $L^{(n)}(x, y)$ contains only terms of order n in x and y , equation (4.5) determines each function $L^{(n)}(x, y)$ uniquely in terms of $f(x)$:

$$L^{(3)} = \frac{1}{2}y(f'(x))^2 \quad (4.8)$$

$$L^{(n+1)} = \int dy \left\{ 1 - \left(1 + \frac{\partial}{\partial y} \sum_{m=3}^n L^{(m)} \right)^2 + \left(f'(x) + \frac{\partial}{\partial x} \sum_{m=3}^n L^{(m)} \right)^2 \right\}.$$

(Keeping only terms of order $n + 1$ in x, y).

This confirms the fact that there are a function of one variable's worth of Lagrangians admitting duality rotations and gives an explicit algorithm for their construction. This method of construction guarantees that

$$L = y + f(x) + \sum_{m=3}^n L^{(m)}(x, y) \quad (4.9)$$

satisfies (4.5) up to and including terms of order $n - 1$ in x and y and hence leads to dual invariant equations of motion to that order.

Exact solutions of the differential equation (3.7) are, in practice, hard to find. One such solution, with $C \neq 0$, may be obtained by separation of variables, using the variables $E + B$ and $E - B$. This procedure gives

$$\begin{aligned} L &= \frac{1}{2}K(B+E)\sqrt{1-\frac{(B+E)^2}{4K^2}}+K^2\sin^{-1}\left(\frac{B+E}{2K}\right) \\ &+ \frac{1}{2}\sqrt{K^2-C}(B-E)\sqrt{1-\frac{(B-E)^2}{4(K^2-C)}} \\ &+ (K^2-C)\sin^{-1}\left(\frac{B-E}{2\sqrt{K^2-C}}\right), \end{aligned} \quad (4.10)$$

where K is an arbitrary constant. However, this solution is pathological since the \mathbf{D} and \mathbf{H} fields defined by

$$\mathbf{D} = \frac{\partial L}{\partial \mathbf{E}} \quad , \quad \mathbf{H} = -\frac{\partial L}{\partial \mathbf{B}} \quad (4.11)$$

are ill-defined at $\mathbf{E} = \mathbf{B} = 0$, since the limit of \mathbf{D} and \mathbf{H} as \mathbf{E} and \mathbf{B} are taken to zero is dependant on the way in which the limit is taken.

5 Other Dualities

One may ask if there are Lagrangians leading to other more general electric-magnetic dualities. The $SO(2)$ rotation (1.11) may be generalized to an $SL(2, \mathbb{R})$ duality transformation. Infinitesimally this takes the form

$$\begin{cases} \delta F_{\mu\nu} = \alpha F_{\mu\nu} + \beta \star G_{\mu\nu} \\ \delta G_{\mu\nu} = -\alpha G_{\mu\nu} - \gamma \star F_{\mu\nu}. \end{cases} \quad (5.1)$$

In terms of \mathbf{E} , \mathbf{B} , \mathbf{D} and \mathbf{H} this gives

$$\begin{cases} \delta \mathbf{E} = \alpha \mathbf{E} - \beta \mathbf{H} \\ \delta \mathbf{B} = \alpha \mathbf{B} + \beta \mathbf{D} \\ \delta \mathbf{D} = -\alpha \mathbf{D} - \gamma \mathbf{B} \\ \delta \mathbf{H} = -\alpha \mathbf{H} + \gamma \mathbf{E}. \end{cases} \quad (5.2)$$

As before, one can work in a Lorentz frame in which \mathbf{E} , \mathbf{B} , \mathbf{D} and \mathbf{H} are all parallel. The invariance of the constitutive relation then gives a pair of second order partial differential constraints on L :

$$\begin{aligned} \frac{\partial}{\partial E} \left(\alpha E \frac{\partial L}{\partial E} + \beta \frac{\partial L}{\partial E} \frac{\partial L}{\partial B} \right) + \alpha B \frac{\partial^2 L}{\partial E \partial B} + \gamma B &= 0 \\ \frac{\partial}{\partial B} \left(\alpha B \frac{\partial L}{\partial B} + \beta \frac{\partial L}{\partial B} \frac{\partial L}{\partial E} \right) + \alpha E \frac{\partial^2 L}{\partial E \partial B} + \gamma E &= 0. \end{aligned} \quad (5.3)$$

However, if one requires that L coincides with Maxwell theory up to and including terms quadratic in E and B , then by expanding L as a power series in E and B and considering terms linear in E and B in equation (5.3), it can be shown that no solutions exist unless $\alpha = 0$ and $\beta = \gamma$. The duality transformation then reduces to the $SO(2)$ rotations considered above.

Therefore, restricted to Lagrangians which coincide with the Maxwell Lagrangian for small values of \mathbf{E} and \mathbf{B} , the only continuous electric-magnetic duality transformations which leave the constitutive relation invariant are $SO(2)$ rotations of the form (1.11). Lagrangians which give rise to more general $SL(2, \mathbb{R})$ dualities cannot coincide with the Maxwell Lagrangian for small \mathbf{E} and \mathbf{B} and must be of the form

$$L = \lambda (\mathbf{E}^2 - \mathbf{B}^2) + \theta \mathbf{E} \cdot \mathbf{B} + \mathcal{O}(4), \quad (5.4)$$

since there is no invariant of $F_{\mu\nu}$ of odd order in the fields. However λ can be made equal to $\frac{1}{2}$ and the θ -term made to vanish by a canonical coordinate transformation (rescaling \mathbf{E} and \mathbf{B} and rotating \mathbf{E} into \mathbf{B}). If the Maxwell field is the only field present, it therefore suffices to consider Lagrangians of the form

$$L = \frac{1}{2} (\mathbf{E}^2 - \mathbf{B}^2) + \mathcal{O}(4) \quad (5.5)$$

and duality transformations of the form (1.11).

Alternatively, the duality transformation properties of Lagrangians with θ -terms may be described by a conjugated action of an $SO(2)$ rotation. Consider a Lagrangian \tilde{L} of the form

$$\tilde{L} = L + \theta \mathbf{E} \cdot \mathbf{B} \quad (5.6)$$

where the Lagrangian L admits an ordinary electric-magnetic duality rotation, i.e. $\frac{\partial L}{\partial E} \frac{\partial L}{\partial B} = -BE$. For which values of α, β, γ (if any) does \tilde{L} admit an $SL(2, \mathbb{R})$ duality transformation of the form (5.1), (5.2)? This is equivalent to asking, for which values of α, β, γ does \tilde{L} satisfy equations (5.3)? It is easy to check that, in general, \tilde{L} will only satisfy (5.3) if $\alpha = -\theta\beta$ and $\gamma = \beta(1 + \theta^2)$. The resulting duality may be rewritten as :

$$\begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} \rightarrow S \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} \quad , \quad \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \rightarrow (S^T)^{-1} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \quad , \quad \mathcal{M} \rightarrow (S^T)^{-1} \mathcal{M} S^{-1} \quad (5.7)$$

where $\mathcal{M}(\mathbf{B}, \mathbf{D})$ is the matrix which encodes the constitutive relation :

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} \quad (5.8)$$

and

$$S = \begin{pmatrix} 1 + \theta\alpha & -\alpha(1 + \theta^2) \\ \alpha & 1 - \theta\alpha \end{pmatrix}. \quad (5.9)$$

This duality is a conjugate action of an $SO(2)$ rotation. This may be seen by writing S in the form

$$S = ARA^{-1} \quad (5.10)$$

where

$$A = \begin{pmatrix} 1 & \theta \\ 0 & 1 \end{pmatrix} \quad (5.11)$$

and R is an infinitesimal rotation :

$$R = \begin{pmatrix} 1 & -\alpha \\ \alpha & 1 \end{pmatrix}. \quad (5.12)$$

The full group action may then be reconstructed simply by taking R to be a finite rotation :

$$R = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}, \quad S = \begin{pmatrix} \cos \alpha + \theta \sin \alpha & -(1 + \theta^2) \sin \alpha \\ \sin \alpha & \cos \alpha - \theta \sin \alpha \end{pmatrix}. \quad (5.13)$$

Thus adding a θ -term to a Lagrangian which admits an $SO(2)$ duality rotation merely conjugates the rotation by a matrix A of the form (5.11).

It is also interesting to ask whether a particular Lagrangian leads to a theory with a *discrete* electric-magnetic duality invariance. In the case of Born-Infeld theory, given by the Lagrangian (4.2), the constitutive relation gives \mathbf{D} and \mathbf{H} in terms of \mathbf{E} and \mathbf{B} :

$$\begin{aligned} \mathbf{D} = +\frac{\partial L}{\partial \mathbf{E}} &= \frac{\mathbf{E} + b^2 (\mathbf{E} \cdot \mathbf{B}) \mathbf{B}}{\sqrt{1 + b^2 (\mathbf{B}^2 - \mathbf{E}^2) - b^4 (\mathbf{E} \cdot \mathbf{B})^2}} \\ \mathbf{H} = -\frac{\partial L}{\partial \mathbf{B}} &= \frac{\mathbf{B} - b^2 (\mathbf{E} \cdot \mathbf{B}) \mathbf{E}}{\sqrt{1 + b^2 (\mathbf{B}^2 - \mathbf{E}^2) - b^4 (\mathbf{E} \cdot \mathbf{B})^2}}. \end{aligned} \quad (5.14)$$

These equations can be solved to give \mathbf{E} and \mathbf{H} in terms of \mathbf{D} and \mathbf{B} :

$$\begin{aligned}\mathbf{E} &= \frac{(1+b^2\mathbf{B}^2)\mathbf{D} - b^2(\mathbf{B} \cdot \mathbf{D})\mathbf{B}}{\sqrt{(1+b^2\mathbf{B}^2)(1+b^2\mathbf{D}^2) - b^4(\mathbf{B} \cdot \mathbf{D})^2}} \\ \mathbf{H} &= \frac{(1+b^2\mathbf{D}^2)\mathbf{B} - b^2(\mathbf{B} \cdot \mathbf{D})\mathbf{D}}{\sqrt{(1+b^2\mathbf{B}^2)(1+b^2\mathbf{D}^2) - b^4(\mathbf{B} \cdot \mathbf{D})^2}}.\end{aligned}\tag{5.15}$$

When expressed in this form, it is clear that there is a discrete electric-magnetic duality corresponding to interchanging \mathbf{B} and \mathbf{D} and also interchanging \mathbf{E} and \mathbf{H} . In the Hamiltonian picture, this discrete duality invariance of the constitutive relation may be seen as a mirror symmetry of the Lagrangian submanifold \mathcal{L} under reflections in the 6-dimensional hyper-plane $\mathbf{B} = \mathbf{D}$, $\mathbf{E} = \mathbf{H}$ of the phase space \mathcal{P} .

To determine whether other Lagrangians also admit this discrete duality it is sufficient to consider a power series expansion of the Lagrangian. Assuming that the Lagrangian coincides with the Maxwell Lagrangian for weak fields, the power series will take the form

$$\begin{aligned}L &= \frac{1}{2}(\mathbf{E}^2 - \mathbf{B}^2) + \alpha(\mathbf{E}^2 - \mathbf{B}^2)^2 + \beta(\mathbf{E} \cdot \mathbf{B})^2 \\ &\quad + \gamma(\mathbf{E}^2 - \mathbf{B}^2)^3 + \delta(\mathbf{E}^2 - \mathbf{B}^2)(\mathbf{E} \cdot \mathbf{B})^2 + \mathcal{O}(8).\end{aligned}\tag{5.16}$$

Differentiating with respect to \mathbf{E} and \mathbf{B} to obtain \mathbf{D} and \mathbf{H} one can then solve for \mathbf{E} and \mathbf{H} in terms of \mathbf{B} and \mathbf{D} as above. Then the condition that the constitutive relation be invariant under discrete duality transformations gives the following constraints on the coefficients in the power series expansion of L :

$$\gamma = 4\alpha^2, \quad \delta = 8\alpha\beta - \beta^2.\tag{5.17}$$

It is easy to check that the power series expansion of the Born-Infeld Lagrangian does indeed satisfy these constraints. On the other hand, the logarithmic Lagrangian which has recently aroused interest in the theory of *black points* [7],

$$L = -\frac{1}{4b^2} \log(1 + b^2 F_{\mu\nu} F^{\mu\nu}),\tag{5.18}$$

does not satisfy the constraints (5.17) and so does not have this discrete duality invariance.

6 Extension to $2k$ -Forms in $4k$ Spacetime Dimensions

The analysis above may be repeated in an almost identical fashion for fields given by $2k$ -forms in $4k$ spacetime dimensions where the operator identity $\star\star = -1$ continues to hold. The Lagrangian will be a function of the completely antisymmetric tensor field $F_{\mu_1 \dots \mu_{2k}}$ where the μ_i range from 0 up to $4k - 1$. $G^{\mu_1 \dots \mu_{2k}}$ may be defined by

$$G^{\mu_1 \dots \mu_{2k}} = -(2k)! \frac{\partial L}{\partial F_{\mu_1 \dots \mu_{2k}}}.$$
 (6.1)

In a local orthonormal frame, one may define the electric and magnetic intensities and inductions in the obvious way :

$$\begin{aligned} E_{i_1 \dots i_{2k-1}} &= F_{i_1 \dots i_{2k-1} 0} \\ B_{i_1 \dots i_{2k-1}} &= \frac{1}{(2k)!} \epsilon_{i_1 \dots i_{2k-1} j_1 \dots j_{2k}} F_{j_1 \dots j_{2k}} \\ D_{i_1 \dots i_{2k-1}} &= G_{i_1 \dots i_{2k-1} 0} \\ H_{i_1 \dots i_{2k-1}} &= \frac{1}{(2k)!} \epsilon_{i_1 \dots i_{2k-1} j_1 \dots j_{2k}} G_{j_1 \dots j_{2k}} \end{aligned}$$
 (6.2)

where i_1, i_2, \dots run from 1 up to $4k - 1$ and $\epsilon_{i_1 \dots i_{4k-1}}$ is the Lévi-Civita symbol in $4k - 1$ space. The Bianchi identities are

$$\partial_{[\lambda} F_{\mu_1 \dots \mu_{2k}]} = 0$$
 (6.3)

which are equivalent to

$$\begin{aligned} \frac{\partial}{\partial x_{i_1}} B_{i_1 \dots i_{2k-1}} &= 0 \\ \epsilon_{i_1 \dots i_{2k-1} l m_1 \dots m_{2k-1}} \frac{\partial}{\partial x_l} E_{m_1 \dots m_{2k-1}} &= -\frac{\partial}{\partial t} B_{i_1 \dots i_{2k-1}}. \end{aligned}$$
 (6.4)

The field equations are

$$\partial_{[\lambda} \star G_{\mu_1 \dots \mu_{2k}]} = 0 \quad (6.5)$$

where now the Hodge star operation is defined by

$$\star G_{\mu_1 \dots \mu_{2k}} = \frac{1}{(2k)!} \eta_{\mu_1 \dots \mu_{2k}}^{\nu_1 \dots \nu_{2k}} G_{\nu_1 \dots \nu_{2k}} \quad (6.6)$$

and $\eta_{\mu_1 \dots \mu_{4k}}$ is the covariantly constant volume form in $4k$ dimensional space-time so that $\star \star = -1$. The field equations are equivalent to

$$\begin{aligned} \frac{\partial}{\partial x_{i_1}} D_{i_1 \dots i_{2k-1}} &= 0 \\ \epsilon_{i_1 \dots i_{2k-1} l m_1 \dots m_{2k-1}} \frac{\partial}{\partial x_l} H_{m_1 \dots m_{2k-1}} &= + \frac{\partial}{\partial t} D_{i_1 \dots i_{2k-1}}. \end{aligned} \quad (6.7)$$

Defining $SO(2)$ electric-magnetic duality rotations as before, invariance of the constitutive relation (6.1) under infinitesimal transformations :

$$\begin{cases} \delta F = \star G \\ \delta G = \star F \end{cases} \quad (6.8)$$

gives the second order differential equation

$$\eta^{\mu_1 \dots \mu_{2k} \nu_1 \dots \nu_{2k}} F_{\nu_1 \dots \nu_{2k}} = \eta_{\lambda_1 \dots \lambda_{2k} \sigma_1 \dots \sigma_{2k}} G^{\sigma_1 \dots \sigma_{2k}} \frac{\partial}{\partial F}_{\lambda_1 \dots \lambda_{2k}} \left(-(2k)! \frac{\partial L}{\partial F}_{\mu_1 \dots \mu_{2k}} \right). \quad (6.9)$$

Rearranging this as before gives ³

$$\begin{aligned} \frac{1}{(2k)!} \eta^{\mu_1 \dots \mu_{2k} \nu_1 \dots \nu_{2k}} F_{\nu_1 \dots \nu_{2k}} \\ = \frac{(2k)!}{2} \frac{\partial}{\partial F}_{\mu_1 \dots \mu_{2k}} \left(\eta_{\lambda_1 \dots \lambda_{2k} \sigma_1 \dots \sigma_{2k}} \frac{\partial L}{\partial F}_{\sigma_1 \dots \sigma_{2k}} \frac{\partial L}{\partial F}_{\lambda_1 \dots \lambda_{2k}} \right), \end{aligned} \quad (6.10)$$

³Note that in $2k$ spacetime dimensions, for k odd, the right hand side of equation (6.10) vanishes identically, since $\eta_{\mu_1 \dots \mu_k \nu_1 \dots \nu_k} = -\eta_{\nu_1 \dots \nu_k \mu_1 \dots \mu_k}$ for k odd. Thus duality rotations of the form (1.11) are not possible in $2, 6, 10, \dots$ spacetime dimensions.

using the fact that $\eta_{\lambda_1\dots\lambda_{2k}\sigma_1\dots\sigma_{2k}} = \eta_{\sigma_1\dots\sigma_{2k}\lambda_1\dots\lambda_{2k}}$. This can now be integrated to give the first order differential equation which generalizes (2.5) :

$$\begin{aligned} & \frac{1}{2(2k)!} \eta^{\mu_1\dots\mu_{2k}\nu_1\dots\nu_{2k}} F_{\mu_1\dots\mu_{2k}} F_{\nu_1\dots\nu_{2k}} \\ &= \frac{(2k)!}{2} \eta_{\mu_1\dots\mu_{2k}\nu_1\dots\nu_{2k}} \frac{\partial L}{\partial F}_{\mu_1\dots\mu_{2k}} \frac{\partial L}{\partial F}_{\nu_1\dots\nu_{2k}} + 2C. \end{aligned} \quad (6.11)$$

Using the constitutive relation, this is equivalent to

$$F_{\mu_1\dots\mu_{2k}} \star F^{\mu_1\dots\mu_{2k}} = G_{\mu_1\dots\mu_{2k}} \star G^{\mu_1\dots\mu_{2k}} + 4C. \quad (6.12)$$

In terms of the electric and magnetic intensities $E_{i_1\dots i_{2k-1}}$ and $H_{i_1\dots i_{2k-1}}$ and the electric and magnetic inductions $D_{i_1\dots i_{2k-1}}$ and $B_{i_1\dots i_{2k-1}}$ this implies

$$E_{i_1\dots i_{2k-1}} B_{i_1\dots i_{2k-1}} = D_{i_1\dots i_{2k-1}} H_{i_1\dots i_{2k-1}} + \frac{2}{(2k)!} C. \quad (6.13)$$

One may also obtain these results using the Hamiltonian theory described above. One replaces $\Lambda^2(\mathbb{R}^4)$ by $\Lambda^{2k}(\mathbb{R}^{4k})$ and proceeds in an almost identical fashion. The simplest example of an electric-magnetic duality invariant Lagrangian would then be the obvious analogue of the Born-Infeld Lagrangian in the form (4.2). This resolves the obvious puzzle of how to generalize the determinant in (1.3) to $2k$ -forms for $k > 1$.

7 The Open SuperString Lagrangian

In open superstring theory, loop calculations lead to an effective Lagrangian density which contains a Born-Infeld term [8]–[11] :

$$\mathcal{L} = \sqrt{g} \left\{ R - 2(\nabla\phi)^2 - \frac{1}{12} e^{2\phi} H^2 \right\} + e^{-3\phi} \left\{ \sqrt{g} - \sqrt{\det(g_{\mu\nu} + e^{2\phi} F_{\mu\nu})} \right\}, \quad (7.1)$$

where

$$H = dB + \frac{1}{4} A \wedge F. \quad (7.2)$$

In four spacetime dimensions, the 3-form H may be eliminated in favour of the axion a giving

$$\begin{aligned}\mathcal{L} &= \sqrt{g} \left\{ R - 2(\nabla\phi)^2 - 2e^{-2\phi} (\nabla a)^2 \right\} + \frac{1}{4} a \sqrt{g} F_{\mu\nu} \star F^{\mu\nu} \\ &\quad + e^{-3\phi} \left\{ \sqrt{g} - \sqrt{\det(g_{\mu\nu} + e^{2\phi} F_{\mu\nu})} \right\}.\end{aligned}\tag{7.3}$$

Here we are using units where $b = 2\pi\alpha' = 1$.

First consider the low energy limit. Keeping only terms quadratic in F , the Lagrangian becomes

$$\begin{aligned}L &= \left\{ R - 2(\nabla\phi)^2 - 2e^{-2\phi} (\nabla a)^2 \right\} + \frac{1}{4} a F_{\mu\nu} \star F^{\mu\nu} - \frac{1}{4} e^\phi F_{\mu\nu} F^{\mu\nu} \\ &= \left\{ R - 2(\nabla\phi)^2 - 2e^{-2\phi} (\nabla a)^2 \right\} + a \mathbf{E} \cdot \mathbf{B} + \frac{1}{2} e^\phi (\mathbf{E}^2 - \mathbf{B}^2).\end{aligned}\tag{7.4}$$

The electric induction and magnetic intensity are then given by

$$\mathbf{D} = e^\phi \mathbf{E} + a \mathbf{B} \quad , \quad \mathbf{H} = e^\phi \mathbf{B} - a \mathbf{E}.\tag{7.5}$$

These constitutive relations may be rewritten as

$$\begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} = \underbrace{\begin{pmatrix} e^{-\phi} & -ae^{-\phi} \\ -ae^{-\phi} & e^\phi + a^2 e^{-\phi} \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix}.\tag{7.6}$$

Equation (7.6) is invariant under an action of $SL(2, \mathbb{R})$. To see how this arises, define a complex scalar field λ by

$$\lambda = a + ie^\phi\tag{7.7}$$

and a complex 2-component vector ψ by

$$\psi = \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}.\tag{7.8}$$

Then the matrix \mathcal{M} may be written as

$$\mathcal{M} = \frac{\psi\psi^\dagger + c.c.}{\sqrt{\det(\psi\psi^\dagger + c.c.)}}.\tag{7.9}$$

Chosing the first component of ψ to be 1 fixes the representation of \mathcal{M} . Equation (7.6) is thus clearly invariant under the following action of $SL(2, \mathbb{R})$:

$$\begin{aligned} \psi \rightarrow \psi' &\propto (S^T)^{-1} \psi \quad \Rightarrow \quad \mathcal{M} \rightarrow (S^T)^{-1} \mathcal{M} S^{-1} \\ \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} &\rightarrow S \begin{pmatrix} \mathbf{D} \\ \mathbf{B} \end{pmatrix} \quad , \quad \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix} \rightarrow (S^T)^{-1} \begin{pmatrix} \mathbf{E} \\ \mathbf{H} \end{pmatrix}. \end{aligned} \tag{7.10}$$

If

$$S = \begin{pmatrix} p & q \\ r & s \end{pmatrix} \quad \text{where } ps - qr = 1, \tag{7.11}$$

then the axion and dilaton fields transform under a M\"obius transformation of λ :

$$\lambda \rightarrow \frac{p\lambda + q}{r\lambda + s}. \tag{7.12}$$

The $F_{\mu\nu}$ equations of motion are thus guaranteed to be invariant under this transformation and it is easy to check that the a and ϕ equations are also invariant and so is the energy-momentum tensor.

It is interesting to ask whether any of this duality invariance is preserved when one considers the full Lagrangian (7.3), which may be written as

$$\begin{aligned} L &= R - 2(\nabla\phi)^2 - 2e^{-2\phi}(\nabla a)^2 + a\mathbf{E} \cdot \mathbf{B} \\ &+ e^{-3\phi} \left\{ 1 - \sqrt{1 + e^{4\phi}(\mathbf{B}^2 - \mathbf{E}^2) - e^{8\phi}(\mathbf{E} \cdot \mathbf{B})^2} \right\}. \end{aligned} \tag{7.13}$$

Hence the electric induction and magnetic intensity are

$$\begin{aligned} \mathbf{D} &= \frac{e^\phi \mathbf{E} + e^{5\phi} (\mathbf{E} \cdot \mathbf{B}) \mathbf{B}}{\sqrt{1 + e^{4\phi}(\mathbf{B}^2 - \mathbf{E}^2) - e^{8\phi}(\mathbf{E} \cdot \mathbf{B})^2}} + a\mathbf{B}, \\ \mathbf{H} &= \frac{e^\phi \mathbf{B} - e^{5\phi} (\mathbf{E} \cdot \mathbf{B}) \mathbf{E}}{\sqrt{1 + e^{4\phi}(\mathbf{B}^2 - \mathbf{E}^2) - e^{8\phi}(\mathbf{E} \cdot \mathbf{B})^2}} - a\mathbf{E}. \end{aligned} \tag{7.14}$$

These equations may be solved to give \mathbf{E} and \mathbf{H} in terms of \mathbf{D} and \mathbf{B} :

$$\begin{aligned}\mathbf{E} &= \frac{(1 + e^{4\phi}\mathbf{B}^2)\mathbf{D} - (a + e^{4\phi}\mathbf{B} \cdot \mathbf{D})\mathbf{B}}{e^\phi \sqrt{(1 + e^{2\phi}\mathbf{D}^2)(1 + e^{4\phi}\mathbf{B}^2) - e^{2\phi}(2a\mathbf{B} \cdot \mathbf{D} + e^{4\phi}(\mathbf{B} \cdot \mathbf{D})^2 - a^2\mathbf{B}^2)}}, \\ \mathbf{H} &= \frac{e^{2\phi}(1 + a^2e^{-2\phi} + e^{2\phi}\mathbf{D}^2)\mathbf{B} - (a + e^{4\phi}\mathbf{B} \cdot \mathbf{D})\mathbf{D}}{e^\phi \sqrt{(1 + e^{2\phi}\mathbf{D}^2)(1 + e^{4\phi}\mathbf{B}^2) - e^{2\phi}(2a\mathbf{B} \cdot \mathbf{D} + e^{4\phi}(\mathbf{B} \cdot \mathbf{D})^2 - a^2\mathbf{B}^2)}}.\end{aligned}\quad (7.15)$$

The matrix \mathcal{M} can then be read off from these equations. Note that these equations agree with equations (5.15) on setting $a = \phi = 0$. However, for $a \neq 0$, the discrete electric-magnetic duality invariance that the pure Born-Infeld theory had is no longer apparent.

As was shown above, this theory has an $SL(2, \mathbb{R})$ duality invariance to lowest order in the electric and magnetic fields. This determines how all the fields must transform under duality and, if the full theory has any dualities, they must be a subgroup of the $SL(2, \mathbb{R})$ duality above. Consider the infinitesimal $SL(2, \mathbb{R})$ transformation described by the matrix

$$S = \begin{pmatrix} 1 - \alpha & \beta \\ \gamma & 1 + \alpha \end{pmatrix}. \quad (7.16)$$

Under this transformation, the fields transform according to :

$$\begin{aligned}a &\rightarrow a(1 - 2\alpha) + \beta + \gamma(e^{2\phi} - a^2), \quad e^\phi \rightarrow e^\phi(1 - 2\alpha - 2a\gamma), \\ \mathbf{E} &\rightarrow (1 + \alpha)\mathbf{E} - \gamma\mathbf{H} \quad , \quad \mathbf{H} \rightarrow (1 - \alpha)\mathbf{H} - \beta\mathbf{E}, \\ \mathbf{D} &\rightarrow (1 - \alpha)\mathbf{D} + \beta\mathbf{B} \quad , \quad \mathbf{B} \rightarrow (1 + \alpha)\mathbf{B} + \gamma\mathbf{D}.\end{aligned}\quad (7.17)$$

It can be shown that these transformations are consistent with the constitutive relations (7.14), (7.15) if and only if $\alpha = -a\gamma$, that is, in contrast to the previous case, the matrix S would have to depend on the axion field a . However, in that case, it is easily seen that the axion and dilaton equations of motion,

$$4\nabla^2 a = 8(\nabla a)(\nabla\phi) - e^{2\phi}\mathbf{E} \cdot \mathbf{B} \quad (7.18)$$

and

$$4\nabla^2\phi + 4e^{-2\phi}(\nabla a)^2 = 3e^{-3\phi} - \frac{3e^{-3\phi} + e^\phi(\mathbf{B}^2 - \mathbf{E}^2) + e^{5\phi}(\mathbf{E} \cdot \mathbf{B})^2}{\sqrt{1 + e^{4\phi}(\mathbf{B}^2 - \mathbf{E}^2) - e^{8\phi}(\mathbf{E} \cdot \mathbf{B})^2}}, \quad (7.19)$$

cannot be invariant under these transformations, unless $\alpha = \gamma = 0$.

Another way of seeing that the theory is not duality invariant is to note that the energy-momentum tensor is not invariant under these transformations. In particular, it is easy to see that the energy density $U = T_{00}$ is not invariant. U may be written as a sum of the contribution from the Born-Infeld term and the contribution from the kinetic terms for the axion and dilaton, $U = U_{BI} + U_{ax-dil}$. U_{BI} may be calculated by a Legendre transformation of the electromagnetic part of the Lagrangian :

$$U_{BI} = \mathbf{E} \cdot \mathbf{D} - L_{em} \quad (7.20)$$

where

$$L_{em} = a\mathbf{E} \cdot \mathbf{B} + e^{-3\phi} \left\{ 1 - \sqrt{1 + e^{4\phi} (\mathbf{B}^2 - \mathbf{E}^2) - e^{8\phi} (\mathbf{E} \cdot \mathbf{B})^2} \right\}. \quad (7.21)$$

The result may be written as

$$U_{BI} = e^{-3\phi} \left\{ \sqrt{1 + e^{4\phi} \mathbf{B}^2 + e^{2\phi} (\mathbf{D} - a\mathbf{B})^2 + e^{6\phi} (\mathbf{B} \times \mathbf{D})^2} - 1 \right\}, \quad (7.22)$$

which is invariant under the transformations (7.17) if $\alpha = -a\gamma$.

However, U_{ax-dil} is not invariant. This can be seen by considering the matrix defined in equation (7.6),

$$\mathcal{M} = \begin{pmatrix} e^{-\phi} & -ae^{-\phi} \\ -ae^{-\phi} & e^\phi + a^2 e^{-\phi} \end{pmatrix}. \quad (7.23)$$

Note that this matrix transforms under $SL(2, \mathbb{R})$ according to

$$\mathcal{M} \rightarrow (S^T)^{-1} \mathcal{M} S^{-1}. \quad (7.24)$$

Also note that the kinetic terms for the axion and dilaton may be written in the form

$$-2(\nabla\phi)^2 - e^{-2\phi}(\nabla a)^2 = 2\det(\nabla\mathcal{M}). \quad (7.25)$$

Therefore they are invariant under *constant* $SL(2, \mathbb{R})$ transformations and hence so is U_{ax-dil} . However, the $SL(2, \mathbb{R})$ transformation necessary to keep the constitutive relation invariant (given by $\alpha = -a\gamma$) is not constant and so the kinetic terms and U_{ax-dil} are not invariant. Hence U is not invariant and so such an electric-magnetic duality is not possible.

Thus all of the $SL(2, \mathbb{R})$ duality invariance of the equations of motion of the Lagrangian (7.4) is lost when one considers the full Born-Infeld term of (7.3) (except for the trivial translational invariance of the axion a).

8 Four Dimensional Spacetime Solutions

To illustrate our results in 4 dimensions we consider spherically symmetric gravitating solutions (in the presence of a cosmological term Λ) of non-linear electrodynamics. We neglect the axion and dilaton in this section since no spacetime solutions of the full theory with the axion and dilaton are yet known. Because the procedure for obtaining local solutions has been extensively studied in the literature [12]–[16] we merely quote the results we need, which are easy enough to obtain anyhow. For any such theory, whether dual symmetric or not, it is known that Birhoff’s theorem holds, and so we may assume the solution is static and takes the form :

$$ds^2 = - \left(1 - \frac{2Gm(r)}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2Gm(r)}{r}} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \quad (8.1)$$

with

$$D(r) \equiv G_{tr} = \frac{Q}{4\pi r^2}, \quad B(r) \equiv F_{\theta\phi} = \frac{P}{4\pi} \sin \theta, \quad (8.2)$$

where Q is the electric and P the magnetic charge and where

$$\frac{dm}{dr} = \frac{\Lambda}{2G} r^2 + 4\pi r^2 U(D(r), B(r)). \quad (8.3)$$

The function $U(\mathbf{D}, \mathbf{B})$ is the energy density in the local orthonormal frame aligned with the static Killing vector. $B(r)$ and $D(r)$ are the magnetic and electric inductions in that frame. The energy density may be obtained from the Lagrangian $L = L(\mathbf{E}, \mathbf{B})$ by a Legendre transformation :

$$U = -L + \mathbf{E} \cdot \mathbf{D}, \quad (8.4)$$

where

$$\mathbf{D} = \frac{\partial L}{\partial \mathbf{E}}. \quad (8.5)$$

For a dual-symmetric theory U depends on \mathbf{B} and \mathbf{D} only in the dual invariant combinations $\mathbf{B}^2 + \mathbf{D}^2$ and $\mathbf{B} \times \mathbf{D}$. Thus, for example, in Born-Infeld theory

$$\begin{aligned} U &= \frac{1}{b^2} \left\{ \sqrt{1 + b^2 (\mathbf{D}^2 + \mathbf{B}^2) + b^4 (\mathbf{D} \times \mathbf{B})^2} - 1 \right\} \\ &= \frac{1}{b^2} \left\{ \sqrt{1 + \frac{b^2 (Q^2 + P^2)}{(4\pi r^2)^2}} - 1 \right\}. \end{aligned} \quad (8.6)$$

It is clear from (8.1), (8.3) and (8.6) that in Born-Infeld theory, the gravitational field of a source with electric charge Q and magnetic charge P , depends only on the dual invariant combination $Z^2 = Q^2 + P^2$ and so $g_{\mu\nu}$ is invariant under duality rotations, which send $Q + iP$ into $e^{i\alpha}(Q + iP)$. This is as expected since $T_{\mu\nu}$ is unchanged by duality transformations in dual invariant theories and hence so is the metric. This will not be true if the theory is not dual invariant. In the case of Born-Infeld theory one finds that

$$-g_{00} = 1 - \frac{2GM}{r} - \Lambda \frac{r^2}{3} + \frac{2G}{b^2 r} \int_r^\infty dx \left(\sqrt{(4\pi x^2)^2 + b^2 Z^2} - 4\pi x^2 \right), \quad (8.7)$$

where M is a constant of integration. If one wants an asymptotically flat solution one must set $\Lambda = 0$. In that case M is the ADM mass. It is helpful to rewrite (8.7) as

$$\begin{aligned} -g_{00} = & 1 - \frac{2G}{r} \left(M - \frac{1}{b^2} \int_0^\infty dx \left(\sqrt{(4\pi x^2)^2 + b^2 Z^2} - 4\pi x^2 \right) \right) \\ & - \Lambda \frac{r^2}{3} - \frac{2G}{b^2 r} \int_0^r dx \left(\sqrt{(4\pi x^2)^2 + b^2 Z^2} - 4\pi x^2 \right). \end{aligned} \quad (8.8)$$

From now on we assume that $\Lambda = 0$. Defining M' by

$$M' = M - \underbrace{\frac{1}{b^2} \int_0^\infty dx \left(\sqrt{(4\pi x^2)^2 + b^2 Z^2} - 4\pi x^2 \right)}_{\mathcal{E}}, \quad (8.9)$$

\mathcal{E} has the interpretation of the energy of the electromagnetic field and so $-M'$ may be interpreted as the binding energy. The electromagnetic field energy \mathcal{E} may be re-expressed as

$$\mathcal{E} = \frac{(Q^2 + P^2)^{\frac{3}{4}}}{\sqrt{4\pi b}} \int_0^\infty dy \left(\sqrt{y^4 + 1} - y^2 \right). \quad (8.10)$$

The y integral may then be integrated by parts becoming a standard elliptic integral which may be expressed in terms of Γ functions :

$$\begin{aligned} I &= \int_0^\infty dy \left(\sqrt{1 + y^{-4}} - 1 \right) y^2 \\ &= \frac{2}{3} \int_0^\infty \frac{dy}{\sqrt{y^4 + 1}} \\ &= \frac{\pi^{\frac{3}{2}}}{3\Gamma\left(\frac{3}{4}\right)^2} = 1.23604978\dots \end{aligned} \quad (8.11)$$

The behaviour of the solutions depends on M' and $\frac{2G}{b}|Z|$. If $M' > 0$ there is just one non-degenerate horizon. If $M' = 0$ and $\frac{2G}{b}|Z| \geq 1$ there is also just one non-degenerate horizon. If $M' = 0$ and $\frac{2G}{b}|Z| < 1$ there is no horizon but because at $r = 0$, $g_{rr} \rightarrow 1 - \frac{2G}{b}|Z|$, one has a conical singularity at the origin. This happens because the energy density is proportional to $\frac{1}{r^2}$ for small r . It seems reasonable to regard the solutions with $M' > 0$ as black holes formed by implosion of the solutions with $M' = 0$.

The case $M' < 0$ is more like Reissner-Nordström in that there may be two non-degenerate horizons or one degenerate horizon or no horizons depending on the size of M relative to $|Z|$ and b . Note that the solutions with one or two horizons always have $M > 0$ and $\frac{2G}{b}|Z| > 1$.

The fact that the purely electric solutions with $M' = 0$ and no horizons have conical singularities was pointed out by Einstein and Rosen. It was one of the reasons why Born abandoned his attempt to construct a finite classical theory of electrically charged particles using the Lagrangian (1.3). If he had succeeded he knew that by dual invariance, he would also have succeeded in constructing a finite classical theory of magnetically charged particles. Since he did not believe that magnetic monopoles exist he chose to abandon (1.3) and turn to other Lagrangians for non-linear electrodynamics which are not dual invariant.

Born's original theory may be viewed in modern terms as an attempt to find classical solutions representing electrically charged "elementary states" with sources which have finite self-energy. As we remarked above in a theory admitting duality invariance, even a discrete dual invariance, there are also magnetically charged classical solutions representing elementary states with sources with finite self-energy.

We now close with some speculative remarks. For the Born-Infeld Lagrangian Born's classical elementary states would have mass

$$M \approx \frac{1.236e^{\frac{3}{2}}}{\sqrt{8\pi^2\alpha'}}, \quad (8.12)$$

equating M with \mathcal{E} given by (8.10). Born tried make contact with quantum field theory by identifying the particles going around closed loops which give rise the Euler-Heisenberg Lagrangian [6] with his electrically charged elementary states. Equating the masses gave a consistency condition which turned out to determine a value for the fine structure constant. Apart from

the fact that this did not give a numerically accurate value (it gave $\alpha \approx \frac{1}{82}$) the procedure was not really consistent because the Euler-Heisenberg Lagrangian for spinors, or scalars, does not coincide, even at lowest non-trivial order, with the Born-Infeld Lagrangian. The calculation would look slightly more convincing if one considered a hypermultiplet consisting of a charged fermion and two charged bosons going around the loop because, in this case at least, the Euler-Heisenberg Lagrangian does coincide with the Born-Infeld Lagrangian at lowest order. Equating \mathcal{E} given by (8.10) with the mass M and using (4.3) gives

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{44}, \quad (8.13)$$

for a purely electrically charged state with electric charge e .

In the light of renormalization theory these calculations of the self-energy of electrically charged particles do not seem very convincing. Currently there is much speculation that elementary string states may be related to extreme black holes. Consider a hypermultiplet of extreme charged black holes with $m = \frac{e}{\sqrt{4\pi G}}$. Virtual black holes going round a closed loop would induce corrections to the Maxwell Lagrangian. We do not know precisely what form this effective Lagrangian takes, but it seems reasonable to suppose that, at lowest order at least, it coincides with the Born-Infeld Lagrangian. This is because both the bosonic and the superstring Lagrangians have this property and moreover the behaviour of black holes is classically invariant under duality rotations and appears to be invariant under discrete duality transformations at the semi-classical level [17].

It is therefore of interest to study the extreme electrically charged black holes of Born-Infeld theory. These are the solutions with $M' \leq 0$, $M > 0$ and $\frac{2G}{b}e > 1$. From (8.8) the degenerate horizon of an extreme black hole is located at $r = r_H$, where r_H satisfies

$$r_H - 2GM' = f(r_H) \quad , \quad \left. \frac{df}{dr} \right|_{r=r_H} = 1 \quad (8.14)$$

and

$$f(r) = \frac{2G}{b^2} \int_0^r dx \left(\sqrt{(4\pi x^2)^2 + b^2 e^2} - 4\pi x^2 \right). \quad (8.15)$$

This implies that

$$r_H = \sqrt{\frac{4G^2 e^2 - b^2}{16\pi G}}. \quad (8.16)$$

Using (8.9) and (8.10) then gives the mass M of an extreme electrically charged Born-Infeld black hole, as a function of its charge e and of b :

$$M \approx \frac{1.236e^{\frac{3}{2}}}{\sqrt{4\pi b}} + \frac{1}{2G} (r_H - f(r_H)). \quad (8.17)$$

For $2Ge \gg b$ this agrees with the charge to mass ratio of the extreme Reissner-Nordström black holes of Einstein-Maxwell theory, i.e. $M = \frac{e}{\sqrt{4\pi G}}$. If $2Ge \gtrsim b$ then $M \approx \frac{1.236e}{\sqrt{8\pi G}}$. The numerical factors $\frac{1}{\sqrt{4\pi}}$ and $\frac{1.236}{\sqrt{8\pi}}$ are approximately the same and plotting a graph of M defined by (8.17) against e and b shows that the charge to mass ratio of extreme electrically charged black holes in Born-Infeld theory is always approximately the same as in Einstein-Maxwell theory.

It thus seems reasonable to assume that the effective Lagrangian obtained by considering virtual black holes going round closed loops will also give rise to extreme electrically charged black holes satisfying $M \approx \frac{e}{\sqrt{4\pi G}}$. Consistency with equation (4.3) requires that $b \approx \sqrt{\frac{2}{3}}G$. If one then thinks that such states correspond to open string states, (1.4) implies that

$$2\pi\alpha' \approx \sqrt{\frac{2}{3}}G. \quad (8.18)$$

Thus if this approach is justified, the ratio of α' to G in string theory will be constrained.

It is also interesting to note that exact, electrically charged extreme solutions of Born-Infeld theory have a minimum electric charge given by $e = \frac{b}{2G}$. For these solutions $\frac{M}{e} \approx \frac{1.236}{\sqrt{8\pi G}}$ and equation (4.3) gives $\alpha = \frac{e^2}{4\pi} \approx \frac{1}{44}$ as before. Clearly the precise details of the calculation above cannot be trusted, however, it raises the interesting possibility that the coupling constant and the ratio of α' to G might be determined by consideration of non-perturbative black hole states. It would therefore be of great interest to obtain solutions with non-vanishing axion and dilaton fields.

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